FHSST Authors

The Free High School Science Texts: Textbooks for High School Students Studying the Sciences Mathematics Grades 10 - 12

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Chapter 34

Independent and Dependent Events - Grade 11

34.1 Introduction

In probability theory an event is either independent or dependent. This chapter describes the differences and how each type of event is worked with.

34.2 Definitions

Two events are independent if knowing something about the value of one event does not give any information about the value of the second event. For example, the event of getting a "1" when a die is rolled and the event of getting a "1" the second time it is thrown are independent.

Definition: Independent Events
Two events A and B are independent if when one of them happens, it doesn't affect the other one happening or not.

The probability of two independent events occurring, $P(A \cap B)$, is given by:

$$P(A \cap B) = P(A) \times P(B) \tag{34.1}$$

Worked Example 151: Independent Events

Question: What is the probability of rolling a 1 and then rolling a 6 on a fair die?
Answer
Step 1 : Identify the two events and determine whether the events are independent or not
Event A is rolling a 1 and event B is rolling a 6. Since the outcome of the first event does not affect the outcome of the second event, the events are independent.
Step 2 : Determine the probability of the specific outcomes occurring, for each event
The probability of rolling a 1 is $\frac{1}{6}$ and the probability of rolling a 6 is $\frac{1}{6}$.
Therefore, $P(A) = \frac{1}{6}$ and $P(B) = \frac{1}{6}$.
Step 3 : Use equation 34.1 to determine the probability of the two events occurring together.

$$\begin{aligned} P(A \cap B) &= P(A) \times P(B) \\ &= \frac{1}{6} \times \frac{1}{6} \\ &= \frac{1}{36} \end{aligned}$$

The probability of rolling a 1 and then rolling a 6 on a fair die is $\frac{1}{36}$.

Consequently, two events are dependent if the outcome of the first event affects the outcome of the second event.

Worked Example 152: Dependent Events

Question: A cloth bag has 4 coins, 1 R1 coin, 2 R2 coins and 1 R5 coin. What is the probability of first selecting a R1 coin followed by selecting a R2 coin?
Answer
Step 1 : Identify the two events and determine whether the events are independent or not
Event A is selecting a R1 coin and event B is next selecting a R2. Since the outcome of the first event affects the outcome of the second event (because there are less coins to choose from after the first coin has been selected), the events are dependent.
Step 2 : Determine the probability of the specific outcomes occurring, for each event
The probability of first selecting a R1 coin is $\frac{1}{4}$ and the probability of next selecting a R2 coin is $\frac{2}{3}$ (because after the R1 coin has been selected, there are only three coins to choose from).
Therefore, $P(A) = \frac{1}{4}$ and $P(B) = \frac{2}{3}$.
Step 3 : Use equation 34.1 to determine the probability of the two events occurring together.
The same equation as for independent events are used, but the probabilities are calculated differently.

$$\begin{aligned} P(A \cap B) &= P(A) \times P(B) \\ &= \frac{1}{4} \times \frac{2}{3} \\ &= \frac{2}{12} \\ &= \frac{1}{6} \end{aligned}$$

The probability of first selecting a R1 coin followed by selecting a R2 coin is $\frac{1}{6}$.

34.2.1 Identification of Independent and Dependent Events

Use of a Contingency Table

A two-way contingency table (studied in an earlier grade) can be used to determine whether events are independent or dependent.

Definition: two-way contingency table
A two-way contingency table is used to represent possible outcomes when two events are combined in a statistical analysis.

For example we can draw and analyse a two-way contingency table to solve the following problem.

Worked Example 153: Contingency Tables

Question: A medical trial into the effectiveness of a new medication was carried out. 120 males and 90 females responded. Out of these 50 males and 40 females responded positively to the medication.

1. Was the medication's succes independent of gender? Explain.
2. Give a table for the independent of gender results.

Answer
Step 1 : Draw a contingency table

	Male	Female	Totals
Positive result	50	40	90
No Positive result	70	50	120
Totals	120	90	210

Step 2 : Work out probabilities
P(male).P(positive result)=$\frac{120}{210} = 0.57$
P(female).P(positive result)=$\frac{90}{210} = 0.43$
P(male and positive result)=$\frac{50}{210} = 0.24$
Step 3 : Draw conclusion
P(male and positive result) is the observed probability and P(male).P(positive result) is the expected probability. These two are quite different. So there is no evidence that the medications success is independent of gender.
Step 4 : Gender-independent results
To get gender independence we need the positve results in the same ratio as the gender. The gender ratio is: 120:90, or 4:3, so the number in the male and positive column would have to be $\frac{4}{7}$ of the total number of patients responding positively which gives 22. This leads to the following table:

	Male	Female	Totals
Positive result	22	68	90
No Positive result	98	22	120
Totals	120	90	210

Use of a Venn Diagram

We can also use Venn diagrams to check whether events are dependent or independent.

Definition: Independent events
Events are said to be independent if the result or outcome of the event does not affect the result or outcome of another event. So P(A/C)=P(A), where P(A/C) represents the probability of event A after event C has occured.

Definition: Dependent events
If the outcome of one event is affected by the outcome of another event such that $P(A/C) \neq P(A)$

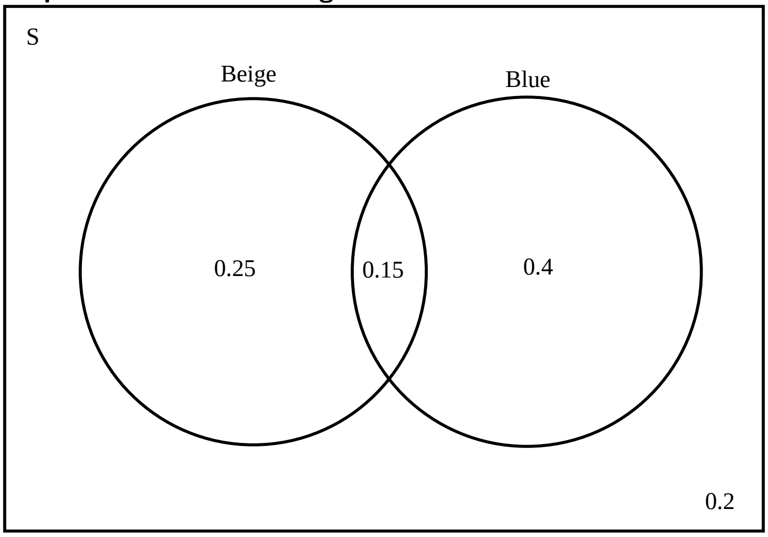
Also note that $P(A/C) = \frac{P(A \cap C)}{P(C)}$ For example, we can draw a Venn diagram and a contingency table to illustrate and analyse the following example.

Worked Example 154: Venn diagrams and events

Question: A school decided that it's uniform needed upgrading. The colours on offer were beige or blue or beige and blue. 40% of the school wanted beige, 55% wanted blue and 15% said a combination would be fine. Are the two events independent?

Answer

Step 1 : Draw a Venn diagram



Step 2 : Draw up a contingency table

	Beige	Not Beige	Totals
Blue	0.15	0.4	0.55
Not Blue	0.25	0.2	0.35
Totals	0.40	0.6	1

Step 3 : Work out the probabilities

P(Blue)=0.4, P(Beige)=0.55, P(Both)=0.15, P(Neither)=0.20
Probability of choosing beige after blue is:

$$\begin{aligned} P(Beige/Blue) &= \frac{P(Beige \cap Blue)}{P(Blue)} \\ &= \frac{0.15}{0.55} \\ &= 0.27 \end{aligned}$$

Step 4 : Solve the problem

Since $P(Beige/Blue) \neq P(Beige)$ the events are statistically independent.

Extension: Applications of Probability Theory

Two major applications of probability theory in everyday life are in risk assessment and in trade on commodity markets. Governments typically apply probability methods in environmental regulation where it is called "pathway analysis", and are often measuring well-being using methods that are stochastic in nature, and choosing projects to undertake based on statistical analyses of their probable effect on the population as a whole. It is not correct to say that statistics are involved in the modelling itself, as typically the assessments of risk are one-time and thus require more fundamental probability models, e.g. "the probability of another 9/11". A law of small numbers tends to apply to all such choices and perception of the effect of such choices, which makes probability measures a political matter.

A good example is the effect of the perceived probability of any widespread Middle East conflict on oil prices - which have ripple effects in the economy as a whole. An assessment by a commodity trade that a war is more likely vs. less likely sends prices up or down, and signals other traders of that opinion. Accordingly, the probabilities are not assessed independently nor necessarily very rationally. The theory of behavioral finance emerged to describe the effect of such groupthink on pricing, on policy, and on peace and conflict.

It can reasonably be said that the discovery of rigorous methods to assess and combine probability assessments has had a profound effect on modern society. A good example is the application of game theory, itself based strictly on probability, to the Cold War and the mutual assured destruction doctrine. Accordingly, it may be of some importance to most citizens to understand how odds and probability assessments are made, and how they contribute to reputations and to decisions, especially in a democracy.

Another significant application of probability theory in everyday life is reliability. Many consumer products, such as automobiles and consumer electronics, utilize reliability theory in the design of the product in order to reduce the probability of failure. The probability of failure is also closely associated with the product's warranty.

34.3 End of Chapter Exercises

1. In each of the following contingency tables give the expected numbers for the events to be perfectly independent and decide if the events are independent or dependent.

		Brown eyes	Not Brown eyes	Totals
A	Black hair	50	30	80
	Red hair	70	80	150
	Totals	120	110	230

		Point A	Point B	Totals
B	Busses left late	15	40	55
	Busses left on time	25	20	35
	Totals	40	60	100

		Durban	Bloemfontein	Totals
C	Liked living there	130	30	160
	Did not like living there	140	200	340
	Totals	270	230	500

		Multivitamin A	Multivitamin B	Totals
D	Improvement in health	400	300	700
	No improvement in health	140	120	260
	Totals	540	420	960

2. A company has a probability of 0.4 of meeting their quota on time and a probability of 0.25 of meeting their quota late. Also there is a 0.10 chance of not meeting their quota on time. Use a Venn diagram and a contingency table to show the information and decide if the events are independent.

3. A study was undertaken to see how many people in Port Elizabeth owned either a Volkswagen or a Toyota. 3% owned both, 25% owned a Toyota and 60% owned a Volkswagen. Draw a contingency table to show all events and decide if car ownership is independent.

4. Jane invested in the stock market. The probability that she will not lose all her money is 1.32. What is the probability that she will lose all her money? Explain.

5. If D and F are mutually exclusive events, with P(D')=0.3 and P(D or F)=0.94, find P(F).

6. A car sales person has pink, lime-green and purple models of car A and purple, orange and multicolour models of car B. One dark night a thief steals a car.

   A What is the experiment and sample space?

   B Draw a Venn diagram to show this.

   C What is the probability of stealing either model A or model B?

   D What is the probability of stealing both model A and model B?

7. Event X's probability is 0.43, Event Y's probability is 0.24. The probability of both occuring together is 0.10. What is the probability that X or Y will occur (this inculdes X and Y occuring simultaneously)?

8. P(H)=0.62, P(J)=0.39 and P(H and J)=0.31. Calculate:

   A P(H')

   B P(H or J)

   C P(H' or J')

   D P(H' or J)

   E P(H' and J')

9. The last ten letters of the alphabet were placed in a hat and people were asked to pick one of them. Event D is picking a vowel, Event E is picking a consonant and Evetn F is picking the last four letters. Calculate the following probabilities:

   A P(F')

   B P(F or D)

   C P(neither E nor F)

   D P(D and E)

   E P(E and F)

   F P(E and D')

10. At Dawnview High there are 400 Grade 12's. 270 do Computer Science, 300 do English and 50 do Typing. All those doing Computer Science do English, 20 take Computer Science and Typing and 35 take English and Typing. Using a Venn diagram calculate the probability that a pupil drawn at random will take:

    A English, but not Typing or Computer Science

    B English but not Typing

    C English and Typing but not Computer Science

    D English or Typing

Appendix A

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